CONVEXITY LATTICES RELATED TO TOPOLOGICAL LATTICES AND INCIDENCE GEOMETRIES

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PREFACE

The idea for this investigation grew out of discussions in a class on multidimensional geometry taught by Professor R. B. Deal. The ultimate objective is that of making a geometric study of manifolds. In the course on multidimensional geometry, incidence geometries are studied by considering properties of a lattice of subsets, called flats, in a manner similar to that used in topology. The purpose of this investigation is to pursue this approach for order and topological structures by studying lattices of subsets called convex sets, properly related to the flats in the incidence geometries, as well as related topologies and uniform structures, although the structures are more general than those of the topological real vector spaces, and each of the classical theorems which hold in these structures must be proved independently. It will be found in this investigation that in most considerations the vector spaces are special cases and the theorems automatically follow.

The ordered pairs refer to the bibliography at the end of the investigation. The first number of the ordered pairs indicates the bibliography reference number; the second refers to the page. For example, (2,15) means the second book of the bibliography, page fifteen.

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CHAPTER I

INTRODUCTION

This investigation is concerned with a study of incidence geometries by considering properties of a lattice of subsets, called flats, and the related convex sets, in a manner similar to that used in topology. The basic properties of the incidence structure were studied in a course entitled Multidimensional Geometry at Oklahoma State University with Professor R. B. Deal. The ultimate goal is an axiomatic study of manifolds. The objectives here are to develop an order structure, to see if any of the classical separation theorems of topological vector spaces can be proved in this setting, to study the related topologies and uniform structures, and to make a brief study of the fundamental properties of mappings in the natural category of general sets with linear and order structures and convex core topologies.

The necessary order structure is developed in Chapter II by making use of half-flats and convex sets. For convenience in making definitions, a list of properties of collections of subsets of a set are enumerated. In a manner similar to that of algebraic structures, some of these involve relations between two or more collections. Some properties of the lattice of convex sets of an incidence geometry are given in Chapter III. Many of the ideas involved are analogous to those found in the theory of vector spaces and topological vector spaces, but here these ideas are discussed and proofs are given without the usual algebraic structure or topology. Theorems which determine when two convex sets can be separated by a hyperflat are fundamental. Most of the work which has been done was done in finite or infinite dimensional vector spaces or topological vector spaces. The goal here is to see how many of the classical separation theorems can be proved for a set with an incidence geometry and the related lattice of convex sets.

Chapter IV is primarily concerned with two types of topologies that can be put on the set S defined in Chapter II. One is the convex core topology which can be introduced directly in a set with an incidence geometry and related order structure with certain properties. This gave too many open sets for many applications. It was decided after many trials at reduction to describe the more general topologies by deriving them directly from a uniformity, with the properties sufficient to relate the topology to the linear and order structures and to provide a locally convex topological vector space with the standard definitions in this case.

Chapter V consists of a brief study of the fundamental properties of mapping in the natural category of general sets with linear and order structures and convex core topologies. The summary is Chapter VI.

CHAPTER II

ORDER STRUCTURE

For convenience in making definitions, the following list of properties of collections of subsets of a set are enumerated. In a manner similar to that of algebraic structure, some of these involve relations between two or more collections.

Throughout this paper, $P(S)$ will denote a generic collection of subsets of S, $L(S)$ a collection of subsets called flats, and $C(S)$ a collection called convex sets, and related to $L(S)$. The properties required of $L(S)$ and $C(S)$ are listed after the enumeration, but each is a complete inclusion lattice with, therefore, a closure operator which will be denoted for $P(S)$, $L(S)$, and $C(S)$, respectively, by π, λ, and μ.

The properties are:

(1) $\phi \in P(S)$,

(2) $S \in P(S)$,

(3) For all $x \in S$, $\{x\} \in P(S)$,

(4) $\mathfrak{M} \subset P(S) \rightarrow \cap \{X \mid X \in \mathfrak{M}\} \in P(S)$,

for $A \subset S$, $\pi(A) = \cap \{X \mid X \in P(S), A \subset X\}$,

(5) $X \in P(S)$, $x \notin X \rightarrow \pi(X \cup \{x\})$ covers X,

(6) $A \subset S$, $\{x,y \in A \rightarrow \pi(x,y) \subset A\} \rightarrow A \in P(S)$,

(7) $A, B \in P(S) \rightarrow A \cup B \in P(S)$,

(8) $P_x(S) = \{X \mid X \in P(S), x \in X\}$ is modular,

(9) For a hyperflat X of Y, $Y \in L(S)$, if $x \in Y \setminus X$ and $\lambda(X \cup \{x\}) = Y$, then $Y \setminus X$ is the union of two disjoint sets A and B of $C(S)$ such that for $a \in A$ and $b \in B$, $\mu(a,b) \cap X \neq \phi$. A and B are called open half-flats of X in Y. $H^{+}(X,x)$ will denote the open half-flat containing x, and $H^{-}(X,x)$ that which does not contain x. Furthermore, $\overline{H}^{\pm}(X,x)$ will denote $H^{\pm}(X,x) \cup X$, and $\overline{H}^{\pm}(X,x) \in C(S)$. Then, $x \notin X \rightarrow \lambda(X \cup \{x\}) = H^{+}(X,x) \cup X \cup H^{-}(X,x)$ and $H^{\pm}(X,x) \cap X = \phi$, $H^{+}(X,x) \cap H^{-}(X,x) = \phi$,

(10) $\mu(x,y) = \overline{H}^{+}(\{x\},y) \cap \overline{H}^{+}(\{y\},x)$,

(11) For $x,y \in S$, $x \neq y \rightarrow \mu(x,y) \neq \{x,y\}$,

(12) For any $x,y \in S$, define ${}^{\bullet}\overline{xy}^{\bullet} = \mu(x,y)$

$$\overline{xy}^{\bullet} = {}^{\bullet}\overline{xy}^{\bullet} \setminus \{x\}$$

$${}^{\bullet}\overline{xy} = {}^{\bullet}\overline{xy}^{\bullet} \setminus \{y\}$$

$$\overline{xy} = {}^{\bullet}\overline{xy} \setminus \{x\}$$

if $x \neq y$, $A \subset \overline{xy}$, $A \neq \phi$, if $U = \{z \mid \overline{zy}^{\bullet} \cap A = \phi\}$ and $\overline{xy} \cap A \neq \phi$, then there exist $u \in \overline{xy}$ such that $t \in \overline{xu} \rightarrow \overline{tu} \cap A \neq \phi$ and $\overline{uy} \cap A = \phi$.

The properties required of $L(S)$ for this paper are: (1), (2), (3), (4), (5), and (6).

$C(S)$ satisfies the following properties: (1), (2), (3), (4), (6), (9), (10), (11), and (12).

The elements of $L(S)$ shall be referred to as L-flats. The L is prefixed because the flats are elements of a lattice which has the additional properties of an incidence geometry. Properties (1) - (6) of $L(S)$ define an incidence geometry. $C(S)$ is a finitely complete lattice, but it is not an incidence geometry. It does not satisfy Property (5).

<u>Theorem 2.1</u>: In a real vector space V the standard collection of flats and convex sets satisfy the requirements for an incidence geometry and related order structure.

<u>Proof</u>: Let $L_V(V) = \{X \mid X \text{ is a translate of a subspace of } V\}$. It should be noted that each subspace of V is an element of $L_V(V)$.

Some properties of $L_V(V)$:

(a) $\phi \in L_V(V)$,

(b) $V \in L_V(V)$,

(c) For all $x \in V$, $\{x\} \in L_V(V)$. $\{x\}$ is a translate of the subspace 0,

(d) $L_V(V)$ is closed under arbitrary intersection,

(e) $X \in L_V(V)$, $x \notin X$, consider the subspace X' of which X is a translate. $\lambda(X',x)$ $x \notin X'$ covers X'; hence, the same translate of $\lambda(X',x)$ covers X,

(f) If $A \subset V$, for any $x,y \in A$ if $\lambda(x,y) \subset A$, then A is a translate of some subspace; therefore, $A \in L_V(V)$.

Properties (a) - (f) define an incidence geometry. The order structure follows from the order structure of R, the real field.

It is clear that L(S) does not have an order structure as one would find in a real vector space, or real topological vector space, but Theorems 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, and 2.8 give the order structure that is essential for this investigation.

<u>Definition 2.1</u>: B(x,y,z) is read y is <u>between</u> x and z, and is defined by B(x,y,z) if and only if $y \in \overline{xz} = \overline{zx}$. It is understood that $x \neq z$.

<u>Theorem 2.2</u>: B(x,y,z) if and only if B(z,y,x).

Proof: $B(x,y,z)$ implies $y \in \overline{xz}$, $\overline{xz} = \overline{zx}$, therefore, $y \in \overline{zx}$; hence, (by Definition 2.1) $B(z,y,x)$. $B(z,y,x)$ implies $y \in \overline{zx}$, $\overline{zx} = \overline{xz}$, therefore, $y \in \overline{xz}$; hence, (by Definition 2.1) $B(x,y,z)$.

Definition 2.2: A point x of a subset B of S is an isolated point of B (with respect to λ) if x is an element of B but not an element of $\lambda(B \setminus \{x\})$. A subset B of S is independent if each point of B is an isolated point.

Theorem 2.3: For any $x,y \in A \subset S$, $x \neq y$ there exists w such that $y \in \overline{xw}$.

Proof: Let $w \in H^-(\{y\},x) \subset \lambda(\{x,y\})$. Now $x \in H^+(\{y\},x) \subset \lambda(\{x,y\})$. By Property (9) of $C(S)$, $H^-(\{y\},x)$, $H^+(\{y\},x) \neq \phi$. Property (9) also implies that $\mu(\{x,w\}) \cap \{y\} \neq \phi$, since y is a hyperflat; $\mu(\{x,w\}) \cap \{y\} = y$. But $x \neq y$, $w \neq y$ so $\overline{xw} \cap y = y$. This proves the theorem.

Theorem 2.4: Let x,y,z be three independent points of S and a line ℓ be a subset of $\lambda(\{x,y,z\})$ such that $[(\ell \cap \overline{xy}) \cup (\ell \cap \overline{yz}) \cup (\ell \cap \overline{zx})] \neq \phi$. If $\{x,y,z\} \cap \ell = \phi$, then exactly two of $\{\ell \cap \overline{xy}, \ell \cap \overline{yz}, \ell \cap \overline{zx}\} \neq \phi$.

Proof: Suppose $\ell \cap \overline{xy} = w \neq \phi$, since $[(\ell \cap \overline{xy}) \cup (\ell \cap \overline{yz}) \cup (\ell \cap \overline{zx})] \neq \phi$. Further suppose $z \in H^+(\ell,x)$, because z is in $H^+(\ell,x)$ or $H^-(\ell,x)$, since $\{x,y,z\} \cap \ell = \phi$. Since $w \in \overline{xy}$, $y \in H^-(\ell,x)$, this implies that $\overline{yz} \cap \ell \neq \phi$. Now $z,x \in H^+(\ell,x)$; therefore, $\ell \cap \overline{xz} = \phi$. If $z \in H^-(\ell,x)$, $x \in H^+(\ell,x)$ by Property (9) of $C(S)$, $\ell \cap \overline{zx} \neq \phi$. And now $y,z \in H^-(\ell,x)$; therefore $\ell \cap \overline{yz} = \phi$. This proves the theorem.

Theorem 2.5: Any four distinct points on a line can be labeled x,y,z,w such that $B(x,y,z)$, $B(x,y,w)$, $B(x,z,w)$, and $B(y,z,w)$.

<u>Lemma 2.1</u>: Any three distinct points on a line can be labeled x,y,z such that B(x,y,z).

<u>Proof</u>: Let a,b,c be three points on a line ℓ. Consider $\overline{ab}$; if $c \in \overline{ab}$, the theorem is proved. Assume $c \notin \overline{ab}$, $c \in H^+(\{b\},a)$ or $c \in H^-(\{b\},a)$; suppose $c \in H^+(\{b\},a)$. Since $c \notin \overline{ab}$, then $a \in \overline{bc}$, and we have B(b,a,c). Relabel b = x, y = a, z = c. If $c \in H^-(\{b\},a)$ and $a \in H^+(\{b\},a)$, by Property (9) of C(S), $\overline{ac} = \mu(\{a,c\}) \cap \{b\} \neq \phi$. Therefore, $\overline{ac} \cap \{b\} = b$, or $b \in \overline{ac}$, and we have B(a,b,c).

<u>Lemma 2.2</u>: Any four distinct points on a line can be labeled x,y,z,w such that y,z,w are in the same half-flat of $\{x\}$.

<u>Proof</u>: Given four points x,y,z,w. Consider three points x,y,z. By Lemma 2.1 they can be labeled such that B(x,y,z). Now there are four possibilities:B(w,x,y), B(x,w,y), B(y,w,z), or B(y,z,w). If B(w,x,y), by Lemma 2.1 w,x,y can be relabeled such that B(x,w,y). Now we have B(x,w,y) and B(w,y,z). In any case, we have B(x,w,y); B(x,y,w) and B(y,w,z) or B(x,y,z) and B(y,z,w), y,z,w are contained in $H^+(\{x\},y)$. This proves the lemma.

<u>Lemma 2.3</u>: If B(x,y,z) and B(y,z,w), then B(x,y,w) and B(x,z,w).

<u>Proof</u>: If B(x,y,z) and B(y,z,w), by Definition 2.1 $y \in \overline{xz}$, $z \in \overline{yw}$, this implies that $z,y \in \overline{xw}$. That is to say that B(x,z,w) and B(x,y,w).

The proof of Theorem 2.4 will now be given.

Let x,y,z,w be any four points on a line. Lemma 2.2 states that x,y,z,w can be labeled such that y,z,w are in the same half-flat of $\{x\}$. That is, $\{y,z,w\} \subseteq H^+(\{x\},z)$. Lemma 2.1 states that y,z,w can be labeled such that $B(y,z,w)$. Lemma 2.3 gives $B(x,y,z)$, $B(x,y,w)$, $B(x,z,w)$, and $B(y,z,w)$. This proves the theorem.

<u>Definition 2.3</u>: For a and b on a line ℓ, $a \neq b$, $\underline{\overleftarrow{a\infty} = \overline{H}^{+}(\{a\},b)}$, $\underline{\overrightarrow{-\infty a} = \overline{H}^{-}(\{a\},b)}$.

<u>Definition 2.4</u>: Given $x,y \in \ell$, $\underline{x \leq y}$ (read x less than or equal to y) if and only if ($x \in \overrightarrow{-\infty a}$ and $y \in \overleftarrow{a\infty}$) or ($x,y \in \overleftarrow{a\infty}$ and $x \in \overline{ay}$) or ($x,y \in \overrightarrow{-\infty a}$ and $y \in \overline{ax}$).

<u>Definition 2.5</u>: Given $x,y \in \ell$, $\underline{x < y}$ (read x less than y) if and only if ($x \in \overline{-\infty a}$ and $y \in \overline{a\infty}$) or ($x,y \in \overline{a\infty}$ and $x \in \overline{ay}$) or ($x,y \in \overline{-\infty a}$ and $y \in \overline{ax}$).

<u>Theorem 2.6</u>: If $x \leq y$ and $y \leq x$, then $x = y$, for $x,y \in \ell$.

<u>Proof</u>: $x \leq y$, by Definition 2.4 there are three cases to be considered.

Case I: $x \leq y$ gives $x \in \overrightarrow{-\infty a}$ and $y \in \overleftarrow{a\infty}$,
$y \leq x$ gives $y \in \overrightarrow{-\infty a}$ and $x \in \overleftarrow{a\infty}$,
this implies $x,y \in \overrightarrow{-\infty a}$ and $x,y \in \overleftarrow{a\infty}$.
Since $\overrightarrow{-\infty a} \cap \overleftarrow{a\infty} = \{a\}$, $x = a = y$.

Case II: $x \leq y$ gives $x,y \in \overleftarrow{a\infty}$ and $x \in \overline{ay}$,
$y \leq x$ gives $x,y \in \overleftarrow{a\infty}$ and $y \in ax$.
These two statements give $x \in \overrightarrow{ay}$ and
$y \in \overrightarrow{ax}$; this implies that $x = y$.

Case III: $x \leq y$ gives $x,y \in \overrightarrow{-\infty a}$ and $y \in \overrightarrow{ax}$,
$y \leq x$ gives $x,y \in \overrightarrow{-\infty a}$ and $x \in \overrightarrow{ay}$.
These two statements give $y \in \overrightarrow{ax}$ and
$x \in \overrightarrow{ay}$; this implies that $x = y$.

<u>Theorem 2.7</u>: If $x \leq y$ and $y \leq z$, then $x \leq z$, for $x, y, z \in \ell$.

<u>Proof</u>: There are three cases to be considered.

Case I. For $x \in -\overrightarrow{\infty a}$, $y \in \overleftarrow{a\infty}$, since $y \in \overleftarrow{a\infty}$, by Definition 2.4, $y \in \overrightarrow{az}(y \leq z)$, which implies $z \in \overleftarrow{a\infty}$. By Definition 2.4, $x \leq z$.

Case II. For $x,y \in \overleftarrow{a\infty}$ and $x \in \overrightarrow{ay}(x \leq y)$, since $y \in \overleftarrow{a\infty}$, then $y \in \overrightarrow{az}(y \leq z)$, $x \in \overrightarrow{ay}$ and $y \in \overrightarrow{az}$ give $x \in \overrightarrow{az}$. By Definition 2.4, $x \leq z$.

Case III. For $x,y \in -\overrightarrow{\infty a}$ and $y \in \overrightarrow{ax}$, since $y \in -\overrightarrow{\infty a}$, then $z \in \overrightarrow{ay}$, $y \in \overrightarrow{ax}$ and $z \in \overrightarrow{ay}$ imply $z \in \overrightarrow{ax}$. By Definition 2.4, $x \leq z$.

<u>Theorem 2.8</u>: For $x \in \ell$, $x \leq x$.

<u>Proof</u>: If $x \in \overleftarrow{a\infty}$, then $x \in \overrightarrow{ax}$, which implies $x \leq x$, by Definition 2.4. If $x \in -\overrightarrow{\infty a}$, then $x \in \overrightarrow{ax}$, which implies $x \leq x$, by Definition 2.4.

CHAPTER III

SOME PROPERTIES OF AN INCIDENCE GEOMETRY AND THE RELATED LATTICE OF CONVEX SETS

Theorems which determine when two convex sets can be separated by a hyperflat are fundamental. Most of the work which has been done was done in finite or infinite dimensional vector spaces or topological vector spaces. The goal here is to see how many of the classical separation theorems can be proved where the spaces used are an incidence geometry and the lattice of convex sets that is compatible with the incidence geometry.

Definition 3.1: An element x of $C \in C(S)$ is called a core point of C if for all $y \in S$, $y \neq x$ there exists a $z \in \overline{xy}$ such that $\overline{xz} \subseteq C$.

Definition 3.2: An element C of C(S) is L-open if all its elements are core points.

Definition 3.3: An element x of S is said to be L-accessible from C an element of C(S) if there exists an element y of C such that $\overline{xy} \subseteq C$.

Definition 3.4: Let L-lina C equal the set of all elements y that are L-accessible from C. By definition, L-lin C = C $\cup$ (L-lina C).

Definition 3.5: A set $D \in C(S)$ is called L-closed if D = L-lin D.

The above definitions were used because they did not make use of a topology or algebraic structure. It should be observed that if the set S were a topological linear space then the core of a convex set C equals its interior and L-lina C = L-lin C = closure of C.

Theorem 3.1 states that the convex closure of two points x,y of S is contained in the closure of x and y in the incidence geometry L(S), and the closure in the incidence geometry L(S) is always convex.

Theorem 3.1: Let x and y be any two elements of a subset A of $Y \in L(S)$; then $\mu(\{x\} \cup \{y\}) \subseteq \lambda(\{x\} \cup \{y\})$.

Proof: By Property (6) of L(S), $x \in \lambda(\{x\} \cup \{y\})$, $y \in \lambda(\{x\} \cup \{y\})$, $\lambda(\{x\} \cup \{y\}) \in C(S)$. By Property (4) of C(S), $\mu(\{x\} \cup \{y\}) \subseteq \lambda(\{x\} \cup \{y\})$. This completes the proof.

Definition 3.6: A hyperflat X of $Y \in L(S)$ strictly separates two disjoint nonempty subsets A and B of Y if for $x \in A$ implies $A \subseteq H^+(X,x)$ and $B \subseteq H^-(X,x)$.

Definition 3.7: A hyperflat X of $Y \in L(S)$ separates two disjoint nonempty subsets A and B of Y if for $x \in A$ implies $A \subseteq [H^+(X,x) \cup X]$, $B \subseteq [H^-(X,x) \cup X]$.

Definition 3.8: A hyperflat F of $Y \in L(S)$ bounds a set $B \subset Y$ if $B \subseteq [H^+(F,x) \cup F]$, $x \in B$.

Definition 3.9: The L-closure of a subset A of S is equal

$$\bar{\mathcal{L}}(A) = \cap \{X \mid X \in \mathcal{L}(S),\ A \subseteq X\}$$

$$\mathcal{L}(S) = \{X \mid X \text{ is L-closed}\}$$

Definitions 3.6, 3.7, and 3.8 are used in this investigation since use could be made of half-flats in place of a linear functional.

Theorems 3.2 and 3.3 are to be expected. Theorem 3.4, 3.5, and 3.6 were needed to prove Theorem 3.7. Theorems 3.7, 3.8, 3.9, and 3.10 are basic to this investigation.

Theorem 3.2: The intersection of any collection of L-closed sets is L-closed.

Proof: Let $U = \cap X_\alpha$ (α element of some index set) each X_α L-closed. Let x be an L-accessible point of U; there exists $y \in U$ such that $\overline{xy} \subset U$; this implies that $\overline{xy} \subset X_\alpha$ for each α. Since each X_α is L-closed, $x \in X_\alpha$ for each α. Therefore, $x \in U$; and by the definition of an L-closed set, U is L-closed.

Theorem 3.3: If A is L-closed, then $\overline{\mathcal{L}}(A) = A$.

Proof: $\overline{\mathcal{L}}(A) = \cap \{X \mid X \in \mathcal{L}(S), A \subset X\}$. Since A is L-closed, then $A \in \mathcal{L}(S)$, and $A \subseteq A$, which imply $\overline{\mathcal{L}}(A) \subseteq A$. The definition of $\overline{\mathcal{L}}(A)$ implies $A \subseteq \overline{\mathcal{L}}(A)$. Therefore, $\overline{\mathcal{L}}(A) = A$.

Theorem 3.4: Let a,b,c be three distinct points of $C \in L(S)$ $a \notin \overline{bc}$, $c \notin \overline{ab}$, given $e \in \overline{ab}$ and $u \in \overline{ce}$, then there exists $w \in \overline{cb}$ such that $u \in \overline{aw}$.

Proof: Consider a,u; by Theorem 2.3, there exists a point p such that $B(a,u,p)$ and $p \in H^-(L,a)$, $(L = \lambda(c,b))$, $p \in H^-(L,a)$ and $a \in H^+(L,a)$. By Property 9 of C(S), there exists $w \in \overline{cb}$ such that $\overline{up} \cap \overline{cb} = w$; w is the desired point.

Theorem 3.5: If a,b,c are three distinct points of $Y \in L(S)$, $a \notin \overline{bc}$, $b \notin \overline{ac}$ and if $e \in \overline{ac}$, $x \in \overline{bc}$, then $\overline{eb} \cap \overline{ax} \neq \phi$.

<u>Proof</u>: Consider $\lambda(e,b) = L$; $\overset{+}{H}(L,x)$, $x \notin L$; $\overset{+}{H}(L,x)$ is the half-space of L containing x. Since $e \in \overline{ac}$, $x \in \overline{bc}$, then $a \in \overset{-}{H}(L,x)$. By Property (9) of C(S), $\overline{ax} \cap L \neq \phi$. That is, $\overline{ax} \cap \overline{be} \neq \phi$.

<u>Theorem 3.6</u>: If C is a convex set of L(S) and $p \in L(S)$ not in C, then $\mu(C \cup \{p\}) = \cup \{\mu(\{c\} \cup \{p\}) \mid c \in C\}$.

<u>Proof</u>: Let $\mu(C \cup \{p\}) = A$, $\cup \{\mu(\{c\} \cup \{p\}) \mid c \in C\} = B$. By Property (4) of C(S), $A \in C(S)$. By the definition of a convex set, $B \subseteq A$. Choose $x_1 \in B$, $x_2 \in B$; there exist c_1 and c_2 elements of C such that $x_1 \in \overleftrightarrow{c_1p}$, $x_2 \in \overleftrightarrow{c_2p}$, and $\overleftrightarrow{c_1c_2} \subseteq C$, since C is convex. Choose any $x \in \overline{x_1x_2}$; there exists $y \in \overline{c_1c_2}$ such that $x \in \overline{yp}$, $\overline{yp} \subset B$, since $y \in C$. Therefore, $x \in B$. This implies $\overleftrightarrow{x_1x_2} \subseteq B$, since x was any point of $\overline{x_1x_2}$. Hence, B is convex. B is convex and contains C and p. By the definition of $\mu(C \cup \{p\}) = A$, $A \subseteq B$. Therefore, $A = B$. This proves the theorem.

<u>Definition 3.10</u>: The sets $A \subseteq Y \in L(S)$, $B \subseteq Y \in L(S)$ are <u>complementary</u> if $A \cup B = Y$ and $A \cap B = \phi$.

<u>Theorem 3.7</u>: If A and B are two disjoint nonempty convex sets of $Y \in L(S)$, then there exist two complementary convex sets C and D such that $A \subseteq C$ and $B \subseteq D$.

<u>Proof</u>: Let P be the set of pairs (A_i,B_i) such that $A \subseteq A_i$, $B \subseteq B_i$, $i \in J$ an ordered index set. Now P is nonempty because A and B belong to P. Partial order P by $(A_i,B_i) \leq (A_j,B_j)$ if $A_i \subseteq A_j$, $B_i \subseteq B_j$. Let P' be a partially ordered chain of P. An upper bound of P' is $(\cup A_i, \cup B_i)$, where (A_i,B_i) is an element of P'. It is easily seen that $(\cup A_i, \cup B_i)$ is an element of P if we can show that $(\cup A_i) \cap (\cup B_i) = \phi$. To see this, let x be an element of $(\cup A_i) \cap (\cup B_i)$. This implies that there exist

an A_i and a B_k such that x is an element of A_i and x is an element of B_k. Now $i \leq k$ or $k \leq i$. Choose $i \leq k$ (in either case the proof would be the same); then x is an element of A_k, since A_k contains A_i. Hence, x is an element of A_k and B_k. This is a contradiction, since A_k and B_k are disjoint by the way P is defined. By Zorn's Lemma, there exists a maximum element (C,D). The proof will be complete if we can show that $C \cup D = Y$. Choose $p \in [Y - (C \cup D)]$ and consider $\mu(C \cup \{p\})$ and $\mu(D \cup \{p\})$. There exist $c_1 \in C \cap [\mu(D \cup \{p\})]$ and $d_1 \in D \cap [\mu(C \cup \{p\})]$, since $\mu(D \cup \{p\})$ and $\mu(C \cup \{p\})$ are convex and (C,D) maximum, $c_1 \notin D$, $d_1 \notin C$ because $C \cap D = \emptyset$. Now there exist $c \in C$, $d \in D$ such that $c_1 \in \overline{dp}$, $d_1 \in \overline{cp}$, because by Theorem 3.6, $\mu(C \cup \{p\}) = \cup \{\mu(\{c\} \cup \{p\}) \mid c \in C\}$ and $\mu(D \cup \{p\}) = \cup \{\mu(d \cup p) \mid d \in D\}$, $\overline{c_1 c} \subset C$, $\overline{d_1 d} \subseteq D$, by Theorem 3.5 $\overline{c_1 c} \cap \overline{d_1 d} \neq \emptyset$. Therefore, $C \cap D \neq \emptyset$ a contradiction. Hence, $C \cup D = Y$.

<u>Theorem 3.8</u>: If A is a convex subset of L(S), then the core of A is convex.

<u>Proof</u>: If the core of A is empty or consists of a single point, the proof is complete, since both are convex. Let x,y be any two elements of the core of A, and $z \in S$, $z \neq x$, $z \neq y$. By the definition of a core point, there exist $w \in \overline{xz}$ and $t \in \overline{yz}$ such that $\overleftrightarrow{wx}$ and $\overleftrightarrow{ty}$ are subsets of A, also $\mu(w,t) \subset A$. We have B(x,w,z) and B(y,t,z). Choose any $p \in \mu(x,y)$ and consider $\overline{zp} \cap \overline{wt} \neq \emptyset$ by Property (9) of C(S). Let $u \in \overline{zp} \cap \overline{wt} \subset A$, and since $p \in A$, then $\overline{up} \subset A$ and p is a core point of A. Therefore, since p was any point in $\overline{xy}$, the core of A is convex.

<u>Theorem 3.9</u>: If C is a convex set of $Y \in L(S)$, $x \in C$ and $y \in$ core of C, then $\overline{xy} \subset$ core of C.

Proof: Let u be any point of $Y \in L(S)$; there exists $w \in \overline{uy}$ such that $\overline{wy} \subset C$ since $y \in$ core of C, $\overline{wx} \subset C$ since C is convex. Let $z \in \overline{xy}$ by Theorem 3.5 $\overline{zu} \cap \overline{xw} \neq \phi$. Choose $z_1 \in \overline{zu} \cap \overline{xw}$, since $z_1 \in C$, $z \in C$, and since C is convex, $\overline{z_1 z} \subset C$. Hence, since u was any point of Y and z any point of $\overline{xy}$, $\overline{xy} \subset$ core of C. This completes the proof.

Theorem 3.10: If $C \in C(S)$, $C \neq \phi$, or $\{x\}$, then every $z \in C$ is an L-accessible point.

Proof: Let $x \in C$; there exists $z \in C$, $z \neq x$; $C \neq \phi$, $C \neq \{x\}$ imply $\overline{zx} \subset C$, since C is convex. Hence, by definition z,x are L-accessible points of C.

The results of Theorems 3.11 and 3.12 are somewhat surprising, since these results do not hold for sets in general.

Theorem 3.11: If C and D are two nonempty complementary convex sets of $Y \in L(S)$, and if $V = \overline{\mathcal{L}}(C) \cap \overline{\mathcal{L}}(D)$, then each point of $C\setminus V$ and $D\setminus V$ is a core point.

Proof: By Theorem 3.13, V is convex; by Theorem 3.2, V is L-closed. Let $z \in C\setminus V$, $x \in Y$, $x \in C\setminus V$ or $x \in V$ or $x \in D\setminus V$. If $x \in V$ and $\overline{xz} \cap V \neq \phi$, then there exists $w \in \overline{xz}$ such that $w \notin V$. If not, then z would be an L-accessible point of V which would contradict V being L-closed, since $z \notin V$. There does not exist $p \in \overline{wz}$ such that $p \in V$. If so, then $w \in V$, since V is convex and $x \in V$. Therefore, $\overline{wz} \subset C\setminus V$ and z is a core point of $C\setminus V$. If $\overline{xz} \cap V = \phi$, then $\overline{xz} \subset C\setminus V$; hence, z is a core point of $C\setminus V$. If $x \in C\setminus V$ and $\overline{xz} \cap V \neq \phi$ by the same reason as above, there exists $w_1 \in \overline{xz}$ such that $\overline{w_1 z} \subset C\setminus V$ and z is a core point of $C\setminus V$. If $x \in D\setminus V$ consider $w_2 \in \overline{zx} \cap V$, if $\overline{w_2 z} \cap V \neq \phi$ by the same reason above, there exists $w_3 \in \overline{w_2 z}$ such that $\overline{w_3 z} \subset C \setminus V$ and z is a core point of $C \setminus V$. Similarly, if $z \in D \setminus V$. This proves the theorem.

<u>Theorem 3.12</u>: If C and D are two nonempty complementary convex sets of $X \in L(S)$ and if $V = \bar{\ell}(C) \cap \bar{\ell}(D)$, then the (core of C) $\cap V = \phi$, and $C \setminus V$ is convex.

<u>Proof</u>: Suppose $V \cap B \neq \phi$ (B = core of C). Let $x \in V \cap B$; since x is an L-accessible point of D, there exists $y \in D$ such that $\overline{xy} \subset D$, and there exists $w \in \overline{xy}$ such that $\overleftrightarrow{xw} \subset C$, since x is a core point of C. This contradicts $C \cap D = \phi$, since $\overleftrightarrow{xw} \subset C$ and $\overline{xy} \subset D$ and $w \in \overline{xy}$. Therefore, $V \cap B = \phi$.

Part II: By the above part of the proof, $V \cap B = \phi$. Therefore, the core of $C = C \setminus V$. By Theorem 3.8, $C \setminus V$ is convex.

<u>Theorem 3.13</u>: If $A \in C(S)$, then $\bar{\ell}(A) \in C(S)$.

<u>Proof</u>: Let $x \in \bar{\ell}(A)$, $y \in \bar{\ell}(A)$ and choose any $z \in \overline{xy}$. By definition of an L-accessible point, there exist $w \in A$ and $t \in A$ such that $\overline{wx} \subset A$, $\overline{ty} \subset A$. Since A is convex if $s \in \overleftrightarrow{wt}$, then $\overline{sz} \subset A$. This can be seen because for $r \in \overline{sz}$, there exist $p \in \overline{xw}$ and $q \in \overline{ty}$ such that $B(p,r,q)$, since $p \in A$, $q \in A$ $\overline{pq} \subset A$, and r was any point in $\overline{sz}$. Therefore, $\overline{sz} \subset A$ and by definition z is an L-accessible point of A. Hence $\overleftrightarrow{xy} \subset \bar{\ell}(A)$ and $\bar{\ell}(A)$ is convex.

<u>Theorem 3.14</u>: If $A \in C(S)$, then $\bar{\ell}(A) = \cap \{X \mid X \in C(S), \bar{\ell}(X) = X, A \subset X\}$.

<u>Proof</u>: Let $\cap \{X \mid X \in C(S), A \subset X, \bar{\ell}(X) = X\} = B$, $\bar{\ell}(A) \in C(S)$, by Theorem 3.13; $\bar{\ell}(A)$ is L-closed by Theorem 3.3. Therefore, $B \subseteq \bar{\ell}(A)$. $\bar{\ell}(A) = \cap \{Y \mid \bar{\ell}(Y) = Y, A \subseteq Y\}$, B is L-closed by Theorem 3.2. Hence,

$\bar{\ell}(A) \subseteq B$. Therefore, $\bar{\ell}(A) = B$. This completes the proof.

<u>Theorem 3.15</u>: If C and D are two nonempty complementary convex sets of $Y \epsilon L(S)$ and if $x \epsilon C$, $y \epsilon D$, $\overline{xy} \cap C = S$, $\overline{xy} \cap D = B$, then $\bar{\ell}(S) \cap \bar{\ell}(B) \neq \phi$.

<u>Proof</u>: S and B are convex; they are the intersection of two convex sets. Let $U = \{z \mid \overline{zy} \cap S = \phi\} = B$ in this case. $S \neq \phi$, $B \neq \phi$ and $S \cup B = \overline{xy}$. By Property (12) of C(S), there exists $u \epsilon \overline{xy}$ such that for all $t \epsilon \overline{xu}$, $\overline{tu} \cap S \neq \phi$ and $\overline{uy} \cap S = \phi$; this implies $\overline{ux} \cap S \neq \phi$. For $t_1 \epsilon \overline{xu} \cap S$, there exists $z_1 \epsilon \overline{t_1u} \cap S$. Let $w \epsilon \overline{z_1u}$; this implies $w \epsilon \overline{xu}$, which implies there exists $w_1 \epsilon \overline{wu} \cap S$; $w \epsilon B$ implies $\overline{wu} \subset B$, but $w_1 \epsilon \overline{wu} \cap S$. This contradicts $C \cap D = \phi$. Therefore, $w \epsilon S$ and $\overline{z_1u} \subset S$, since w was any point of $\overline{z_1u}$; $z \epsilon S$ implies $u \epsilon \bar{\ell}(S)$; $u \epsilon B$ implies $u \epsilon \bar{\ell}(B)$, which implies $\bar{\ell}(S) \cap \bar{\ell}(B) \neq \phi$. If $u \notin B$, then $u \epsilon S$, which implies $u \epsilon \bar{\ell}(S)$; $\overline{uy} \cap S = \phi$ implies $p \epsilon \overline{uy}$; $\overline{up} \cap S = \phi$, since $\overline{up} \subset \overline{uy}$, which implies $p \epsilon B$, $\overline{uy} \neq \phi$, and $\overline{up} \neq \phi$ by Property (12) of C(S); $p_1 \epsilon \overline{up}$ implies $\overline{up_1} \subseteq \overline{uy}$, which implies $\overline{up_1} \cap S = \phi$. Hence, $p_1 \epsilon B$. Therefore, $\overline{up} \subseteq B$. By definition of an L-accessible point, $u \epsilon \bar{\ell}(B)$. Therefore, $\bar{\ell}(S) \cap \bar{\ell}(B) \neq \phi$. This completes the proof.

Theorems 3.16, 3.17, and 3.18 are more general than their classical counterparts. If L(S) is a real vector space, the classical theorems will follow automatically in most cases.

<u>Theorem 3.16</u>: If C and D are two nonempty complementary convex sets of $Y \epsilon L(S)$ and if $V = \bar{\ell}(C) \cap \bar{\ell}(D)$, then V is a hyperflat or $V = Y$.

<u>Proof</u>: By Theorem 3.13, $\bar{\ell}(C)$ and $\bar{\ell}(D)$ are convex. By Property (4) of C(S), V is convex; by Theorem 3.2, V is L-closed; by Theorem 3.15, $V \neq \phi$.

If $V = \{z\}$, then $V \in L(S)$ by Property (2) of $L(S)$. So we can assume that V consists of more than a single point. Let $x \in V$, $y \in V$ and let z be such that $B(x,y,z)$ or $y \in \overline{xz}$. If no such z exists, then $\lambda(x,y) \subset V$, since V is convex and in this case $\mu(x,y) = \lambda(x,y)$; hence, V is an element of $L(S)$. If such a z does exist, then V is an element of $L(S)$ if we can show that z belongs to V. We can assume that $z \in C\setminus V$ or $z \in D\setminus V$. If $C\setminus V = D\setminus V = \phi$, then $V = Y$. So assume that $C\setminus V$ or $D\setminus V$ is not empty. Suppose $z \in C\setminus V$. By Theorem 3.11, each point of $C\setminus V$ is a core point, and the core of $C = C\setminus V$ by Theorem 3.12. Since x is an accessible point of C, $\overline{xz}$ belongs to the core of C by Theorem 3.9. But by the assumption $y \in \overline{xz}$, y is a core point of $C\setminus V$, a contradiction, since y is an accessible point of $C\setminus V$. Therefore, $z \in V$ and V is a flat. The proof is similar if $B(z,x,y)$.

To see that V is a hyperflat, let $x \in C\setminus V$, $y \in D\setminus V$. By Theorem 3.15, $\overline{xy} \cap V \neq \phi$. Consider $\lambda(V,x) = \overset{+}{H}(V,x) \cup V \cup \overset{-}{H}(V,x)$. We have $\lambda(V,x) \subseteq D\setminus V \cup V \cup C\setminus V$. By Property (9) of $C(S)$, $\overline{x_1y_1} \cap V \neq \phi$, for $x_1 \in \overset{+}{H}(V,x)$, $y_1 \in \overset{-}{H}(V,x)$. Let $z \in \overset{+}{H}(V,x)$; then $z \in D\setminus V$ or $z \in C\setminus V$; $z \in D\setminus V$ implies $V \cap \overset{+}{H}(V,x) \neq \phi$, since $x \in C\setminus V$; $z \in D\setminus V$ implies $\overline{xz} \cap V \neq \phi$ and $\overline{xz} \subset \overset{+}{H}(V,x)$, which is a contradiction. Hence, $z \in C\setminus V$ and $\overset{+}{H}(V,x) \subseteq C\setminus V$. Let $w \in \overset{-}{H}(V,x)$; then $w \in C\setminus V$ or $w \in D\setminus V$; $w \in C\setminus V$ implies $\overline{wx} \subset C\setminus V$, which implies $C\setminus V \cap V \neq \phi$, a contradiction, since $\overline{wx} \cap V \neq \phi$. Hence, $w \in D\setminus V$ and $\overset{-}{H}(V,x) \subseteq D\setminus V$.

Let $p \in D\setminus V$; $u \in (\overline{px} \cap V)$; this implies that $\lambda(u,x) \subset \lambda(V,x)$, since $\lambda(V,x)$ is a flat and $u \in V$ and $x \in \lambda(V,x)$; $u \in \overline{px}$ implies $p \in \lambda(u,x)$; $\lambda(u,x) \subseteq \lambda(V,x)$ implies $p \in \lambda(u,x)$; $p \in \overset{-}{H}(V,x)$, since $D\setminus V \cap C\setminus V = \phi$, and $\overset{+}{H}(V,x) \subseteq C\setminus V$. Hence, $D\setminus V \subseteq \overset{-}{H}(V,x)$. Therefore, $\overset{-}{H}(V,x) = D\setminus V$. Let $s \in C\setminus V$, and $r \in \overset{-}{H}(V,x)$; $t \in \overline{sr} \cap V$ implies $\lambda(t,r) \subset \lambda(V,x)$ and

$s \in \lambda(t,r)$ implies $s \in H^+(V,x)$, since $s \in C\setminus V$, $D\setminus V \cap C\setminus V = \phi$ and $H^-(V,x) \subseteq D\setminus V$. This implies $C\setminus V \subseteq H^+(V,x)$. Therefore, $H^+(V,x) = C\setminus V$. $H^+(V,x) = C\setminus V$; $H^-(V,x) = D\setminus V$ implies $C\setminus V \cup D\setminus V \cup V = H^+(V,x) \cup V \cup H^-(V,x)$. Therefore, V is a hyperflat by Property (9) of C(S).

<u>Theorem 3.17</u>: Suppose A and B are two convex subsets of Y, an element of L(S). Suppose further that the core of $B \neq \phi$, $A \neq \phi$, and that (A $\cap$ core of B) $= \phi$; then there exists a hyperflat V which separates A and B.

<u>Proof</u>: Let B^o = the core of B; by Theorem 3.8, B^o is convex. By Theorem 3.7, there exist complementary convex sets C and D such that $B^o \subseteq C$, $A \subseteq D$. By Theorem 3.17, there exists a hyperflat $V = \overline{\ell}(C) \cap \overline{\ell}(D)$, since $B^o = C\setminus V \neq \phi$, or $B^o = H^-(V,x)$, $x \in A$. Suppose $z \in B$, $z \in H^+(V,x)$. By Theorem 3.11, z is a core point of B, which implies for $y \in B^o$ that $\overline{zy} \subset B^o$, since B^o is convex. But $\overline{yz} \cap V \neq \phi$; this contradicts $B^o \cap V = \phi$. Therefore, there does not exist $z \in B$ and $z \in H^+(V,x)$. Hence, $B \subseteq H^-(V,x) \cup V$. Since $A \subseteq H^+(V,x) \cup V$, V separates A and B.

<u>Theorem 3.18</u>: Let A and B be two convex sets in $Y \in L(S)$ and $A \cap B = \phi$; if A and B both have core points, then there exists a hyperflat V which separates A and B.

<u>Proof</u>: By Theorem 3.7, there exist complementary convex sets C and D such that $B \subseteq C$ and $A \subseteq D$. By Theorem 3.17, there exists a hyperflat $V = \overline{\ell}(C) \cap \overline{\ell}(D)$, since the core of A and B are not empty, $B \subseteq C\setminus V \cup V$ and $A \subseteq D\setminus V \cup V$. By Theorem 3.16, $H^+(V,x) = C\setminus V$, $H^-(V,x) = D\setminus V$. Therefore, V separates A and B.

Definition 3.11: A set C is between sets A and B if for any two elements $a \in A$, $b \in B$ there exists an element $c \in C$ such that $B(b,c,a)$.

Theorem 3.19: If X is a hyperflat of $Y \in L(S)$, there exist disjoint sets A and B elements of $C(S)$ such that X is between A and B.

Proof: By definition of a hyperflat, $Y = H^{+}(X,x) \cup X \cup H^{-}(X,x)$ where $H^{+}(X,x)$, $H^{-}(X,x)$ are elements of $C(S)$ and $H^{+}(X,x) \cap H^{-}(X,x) = \phi$. For $x \in H^{-}(X,x) = A$, $y \in H^{-}(X,x) = B$ implies $\overline{xy} \cap X \neq \phi$ by Property (9) of $C(S)$; $z \in \overline{xy} \cap X$ implies $B(x,z,y)$. Therefore, X is between A and B.

Theorem 3.20: If a hyperflat X of $Y \in L(S)$ strictly separates two disjoint sets A and B, then X is between A and B.

Proof: From the hypothesis $A \subseteq H^{+}(X,x)$, $x \in A$, $B \subseteq H^{-}(X,x)$. For $z \in A$, $y \in B$ and by Property (9) of $C(S)$, $\overline{zy} \cap X \neq \phi$. Let $w \in \overline{zy} \cap X$; then we have $B(z,w,y)$. This completes the proof.

CHAPTER IV

TOPOLOGICAL PROPERTIES

This chapter shall be primarily concerned with two types of topologies that can be put on S. One is the convex core topology which can be introduced directly in a set with an incidence geometry and related order structure with the properties discussed in Chapter II. This gives too many open sets for many of the applications. It was decided after many trials at reduction to describe the more general topologies by deriving them directly from a uniformity, with the properties sufficient to relate the topology to the linear and order structures and to provide a locally convex topological vector space with the standard definitions in the case where S is a real vector space.

Let $F_c = \{A \mid$ for all $x \in S\backslash A$ implies there exists $C \in C(S)$ such that $C \subset S\backslash A$ and x is a core point of $C\}$.

Theorem 4.1: The intersection of any collection of elements of F_c is an element of F_c.

Proof: Let $X = \cap X_\alpha$, $X_\alpha \in F_c$ (α an element of some index set), $x \in S\backslash X$ implies $x \in S\backslash \cap X_\alpha$. $S\backslash \cap X_\alpha = \cup X'_\alpha$ (X'_α is the complement of X'_α relative to S) implies $x \in X'_\alpha$ for some α, that is $x \in S\backslash X_\alpha$ for some α; hence, there exists $C \in C(S)$ such that $x \in C \subseteq S\backslash X_\alpha$, and x is a core point of C; therefore, $X \in F_c$.

<u>Theorem 4.2</u>: If A_1 and A_2 are elements of F_c, then $A_1 \cup A_2$ is an element of F_c.

<u>Proof</u>: If we choose $x \in S \setminus (A_1 \cup A_2)$, $x \notin (A_1 \cup A_2)$ implies $x \notin A_1$, $x \notin A_2$; therefore, $x \in S \setminus A_1$, $x \in S \setminus A_2$. By definition, there exist C_1, C_2 elements of $C(S)$ such that $x \in C_1 \subseteq S \setminus A_1$, $x \in C_2 \subseteq S \setminus A_2$ and x is a core point of C_1 and C_2. Let $C = C_1 \cap C_2$, $C \in C(S)$, and $x \in C \subset S \setminus (A_1 \cup A_2)$. For all $z \in S$, $z \neq x$ there exists $w_1 \in \overline{zx}$ such that $\overline{w_1x} \subset C_1$ and there exists $w_2 \in \overline{zx}$ such that $\overline{w_2x} \subset C_2$. It is easily seen that $\overline{w_1x} \subset \overline{w_2x}$ or $\overline{w_2x} \subset \overline{w_1x}$, say $\overline{w_1x} \subset \overline{w_2x}$, then $\overline{w_1x} \subset C$. This proves the theorem.

By Theorems 4.1 and 4.2, and since ϕ and S are elements of F_c, S is a topological space with F_c as a topology and the complement of each element of F_c as open sets.

Let $N_x = \{U \mid \text{there exists } V \in C(S) \text{ such that } V \subset U, x \text{ is a core point of } V\}$. S is a real vector space and C(S) the related convex sets.

The collection N_0 is the standard definition of the neighborhood filter of 0 in the convex core topology. It is well known (3,11) that this gives a topological vector space. As a matter of fact, it is the largest locally convex topology on a real vector space; and all linear functions in the algebraic dual are continuous in this topology.

In order to show that the topology defined for the more general setting is equivalent to the convex core topology for real vector spaces, it is sufficient to show that the usual procedure of defining a neighborhood filter $\tilde{N}_x$ by x-translates of N_0 is the same as N_x defined above.

<u>Theorem 4.3</u>: If S is a real vector space with the standard incidence geometry and collection of convex sets, the above topology is the

standard convex core topology.

Proof: Let $U \in N_x$, $V \subset U$ such that V is convex and x is a core point of V. $V - x = V'$, V' is convex and 0 is a core point of V'; hence, $V' \in \tilde{N}_x$.

Let $W \in \tilde{N}_x$, W is convex, and x is a core point of W. Therefore, $W \in N_x$, and $\tilde{N}_x$ and N_x are the same.

Uniformity

Let S be a set defined as in Chapter II, and consider $S \times S$. $\tilde{L}(S) \subseteq 2^{S \times S}$ is an incidence geometry defined by:

(1) $A, B \in L(S) \rightarrow A \times B \in \tilde{L}(S)$

(2) π is a projection map π_x: $S \times S \rightarrow S, \pi_x: (x,y) \rightarrow x$, $\pi_y: (x,y) \rightarrow y$

(3) $X \in L(S),\ x \in S,\ y \in S \rightarrow \{x\} \times X,\ X \times \{y\} \in \tilde{L}(S)$

(4) $X \in \tilde{L}(S) \rightarrow \pi_x(X) \in L(S),\ \pi_y(X) \in L(S)$.

$\tilde{C}(S) \subseteq 2^{S \times S}$ is a collection of convex sets compatible with $\tilde{L}(S)$. $\tilde{C}(S)$ is defined by:

(1) $A, B \in C(S) \rightarrow A \times B \in \tilde{C}(S)$

(2) π is a projection map π_x: $S \times S \rightarrow S, \pi_x: (x,y) \rightarrow x$, $\pi_y: (x,y) \rightarrow y$

(3) $X \in C(S),\ x \in S,\ y \in S \rightarrow \{x\} \times X,\ X \times \{y\} \in \tilde{C}(S)$

(4) $X \in \tilde{C}(S) \rightarrow \pi_x(X) \in C(S),\ \pi_y(X) \in C(S)$.

It should be observed that if S is a real vector space then $\tilde{L}(S) = \{A \mid A \subset S \times S,\ (x,y) \in A,\ (u,v) \in A,\ \alpha \in R \rightarrow (\alpha x + (1 - \alpha)u,\ \alpha y + (1 - \alpha)v) \in A\}$ and $\tilde{C}(S) = \{B \mid B \subseteq S \times S,\ (x,y) \in B,\ (u,v) \in B,$

$0 \leq \alpha \leq 1 \rightarrow (\alpha x + (1 - \alpha)u, \alpha y + (1 - \alpha)v) \in B\}$.

$U[x] = \{y \mid (x,y) \in U\}$, $U \circ V = \{(x,z) \mid$ for some y, $(x,y) \in U$ and $(y,z) \in V\}$, $\Delta = \{(x,x) \mid x \in S\}$, $U^{-1} = \{(y,x) \mid (x,y) \in U\}$.

Let $\tilde{B}$ be a collection of subsets of $2^{S \times S}$ with the following properties:

(1) $\tilde{C} \in \tilde{B}$ implies $\Delta \subset \tilde{C}$, $\tilde{C} = \tilde{C}^{-1}$, $\tilde{C} \in C(S)$.

(2) For all $z \in S$, (z,z) is a core point of $\tilde{C}$.

(3) For all $\tilde{C} \in \tilde{B}$, there exists $\tilde{C}_1 \in \tilde{B}$ such that $\tilde{C}_1 \circ \tilde{C}_1 \subseteq \tilde{C}$.

(4) If $\tilde{C} \subset S \times S$ satisfies (1), (2), and (3), then $\tilde{C} \in \tilde{B}$.

(5) $\cap\, \tilde{C}[x] = \{x\}$, $\tilde{C} \in \tilde{B}$. $\tilde{B}$ is not empty; it contains $S \times S$.

Let $\mathfrak{M} = \{U \mid$ there exists $\tilde{C} \in \tilde{B}$ such that $\tilde{C} \subset U\}$.

Some properties of $\mathfrak{M}$:

(1) Each member of $\mathfrak{M}$ contains the diagonal Δ (Property (1) of $\tilde{B}$),

(2) If $U \in \mathfrak{M}$, then $U^{-1} \in \mathfrak{M}$ (Property (1) of B).

(3) If $U \in \mathfrak{M}$, then there exists $V \in \mathfrak{M}$ such that $V \circ V \subseteq U$ (Property (3) of B),

(4) If U and V are members of $\mathfrak{M}$, then $U \cap V \in \mathfrak{M}$. There is need to consider only elements of $\tilde{B}$. $U \cap V$ contains Δ, since both U and V contain Δ. The intersection of two symmetric sets is symmetric; hence, $U \cap V$ is symmetric, $U \cap V \in \tilde{C}(S)$, since U and V are convex. There exists W such

that $W \circ W \subseteq U \cap V$, since for each U and V there exists V_1 and U_1 such that $V_1 \circ V_1 \subseteq V$, $U_1 \circ U_1 \subseteq U$, choose $W = V_1 \cap U_1$. Consider $W \circ W$, $(x,z) \in W \circ W$ implies there exists $y \in W$ such that (x,y) and (y,z) are elements of W which imply that (x,y) and (y,x) are both elements of V_1 and U_1, which imply (x,y) and (y,z) are contained in both U and V, hence, in $U \cap V$.

(5) If $U \in \mathfrak{M}$ and $U \subset V \subset S \times S$, then $V \in \mathfrak{M}$. This follows from the way $\mathfrak{M}$ is defined.

Properties (1) - (5) of $\mathfrak{M}$ define a uniformity on S (5,176). The pair $(S,\mathfrak{M})$ shall be referred to as a uniform space.

A study shall now be made of the uniform space $(S,\mathfrak{M})$ when S is a vector space over the reals. $(S,\mathfrak{M})$ can be made into a topological space by making use of the uniform topology, that is, by defining a set A to be open if and only if for each $x \in A$ there exists $U \in \mathfrak{M}$ such that $U[x] \subset A$.

<u>Theorem 4.4</u>: In a locally convex topological real vector space, the standard uniformity satisfies the conditions listed above and the topology introduced above is sufficient to insure that a real vector space with the standard incidence geometry and order structure is a locally convex topological vector space.

Proof of Part I of the above theorem is given by Theorem 1.4 (9,160). Part II is given by the following theorems.

For the discussion that follows, S is a real vector space. Let $\mathfrak{N}_0 = \{\widetilde{C}[0] \mid \widetilde{C} \in \widetilde{B}\}$.

<u>Theorem 4.5</u>: If $U \in \mathfrak{N}_o$, then $-U \in \mathfrak{N}_o$.

<u>Proof</u>: Consider the way $\tilde{B}$ is defined. If $(x,x) \in U$, then $(-x,-x) \in -U$, since S is a real vector space; this implies $\Delta \subseteq -U$. $-U$ is convex, since S is a real vector space. Symmetry of $-U$ follows from (x,y), and (y,x) elements of U imply $(-x,-y)$, and $(-y,-x)$ are elements of $-U$. It has been proved that if a point x is a core point of U it is a core point of $-U$. Since there exists $\tilde{C}_1 \in \tilde{B}$ such that $\tilde{C}_1 \circ \tilde{C}_1 \subseteq C$, then $-\tilde{C}_1 \circ -\tilde{C}_1 \subseteq -C$. That is, if $(-x,-z) \in -\tilde{C}_1 \circ -\tilde{C}_1$, there exists $-y \in -\tilde{C}_1$ such that $(-x,-y)$ and $(-y,-z)$ are elements of $-\tilde{C}_1$; hence, (x,y) and (y,z) are elements of $\tilde{C}_1$ and $(x,z) \in \tilde{C}$. Therefore, $(-x,-z) \in -C$. This proves that $-U \in \mathfrak{N}_o$.

Let us consider the collection of subsets of $\mathfrak{N}_o$, $\mathfrak{N}_o' = \{V \mid V = U \cap -U,\ U \in \mathfrak{N}_o\}$. Each $V \in \mathfrak{N}_o'$ is convex, symmetric in the uniformity and algebraically; V is also balanced, since it is symmetric (algebraically) and convex.

Some properties of $\mathfrak{N}_o'$:

(1) Zero is the only point common to all $V \in \mathfrak{N}_o'$. (This follows from Property (5) of $\tilde{B}$.)

(2) If U and V are members of $\mathfrak{N}_o'$, then there exists $W \in \mathfrak{N}_o'$ such that $W \subseteq U \cap V$. (This follows from the fact that $U \cap V \in \mathfrak{N}_o'$.)

(3) If $U \in \mathfrak{N}_o'$, then for $|r| \leq 1$, r, real $rU \subseteq U$, since each U is balanced.

(4) For all $U \in \mathfrak{N}_o'$, there exists $V \in \mathfrak{N}_o'$ such that $V + V \subseteq U$.

(a) If $U \in \mathfrak{N}_o'$, then $rU \in \mathfrak{N}_o'$, $0 < r$, r, real.

Since S is a real vector space, rU is convex. For all $(x,x) \in U$, $(\frac{1}{r}x, \frac{1}{r}x) \in U$; hence, $r(\frac{1}{r}x, \frac{1}{r}x) = (x,x) \in rU$, for all $x \in S$. Therefore, the diagonal Δ is in rU. To see that rU is symmetric, consider (x,y), $(y,x) \in U$ which implies that (rx,ry), $(ry,rx) \in rU$. For all $z \in S$, if $(\frac{1}{r}z, \frac{1}{r}z)$ is a core point of U, it is easily seen that $r(\frac{1}{r}z, \frac{1}{r}z) = (z,z)$ is a core point of rU. It is left to show there exists $V \in \mathfrak{N}'_o$ such that $V \circ V \subseteq rU$. Since $U \in \mathfrak{N}'_o$, there exists $V_1 \in \mathfrak{N}'_o$ such that $V_1 \circ V_1 \subseteq U$, choose $V = rV_1$.

(b) If $U \in \mathfrak{N}'_o$, then $rU + rU = (r + r)U = 2rU$, r a positive real. (This follows from the fact that S is a real vector space.)

To see (4), let $V = \frac{1}{2}U$; then we have $\frac{1}{2}U + \frac{1}{2}U = U \subseteq U$.

A topological vector space is defined by Properties (1) - (4) of $\mathfrak{N}'_o$ (3,12).

CHAPTER V

THE CATEGORY

This chapter contains a brief discussion of the fundamental properties of mappings in the natural category of general sets with linear and order structures and convex core topologies.

Let us consider the category Ω with objects quadruples $(S, L(S), C(S), F_c(S))$ where S, $L(S)$, $C(S)$, and $F_c(S)$ are as defined in Chapters II and IV. The morphisms are the maps f, that is, $\{f \mid f : S_1 \to S_2$ and (1) $f(\lambda_1(A)) = \lambda_2(f(A))$, (2) $f(\mu_1(A)) = \mu_2(f(A))\}$.

<u>Theorem 5.1</u>: If $X_1 \in L_1(S_1)$, then $f(X_1) \in L_2(S_2)$ where f is a morphism on S_1 to S_2 with Properties (1) and (2).

<u>Proof</u>: By Property (1) of f, $f(\lambda_1(X_1)) = \lambda_2(f(X_1))$, $f(\lambda_1(X_1)) = \lambda_2(f(X_1))$ implies $f(X_1) \in L_2(S_2)$, since $f(\lambda_1(X_1)) = f(X_1)$.

<u>Theorem 5.2</u>: If $A_1 \in C_1(S_1)$, then $f(A_1) \in C_2(S_2)$.

<u>Proof</u>: $f(\mu_1(A_1)) = f(A_1)$, since $\mu_1(A_1) = A_1$, $f(A_1) = f(\mu_1(A_1)) = \mu_2(f(A_1))$ by Property (2) of f. Therefore, $f(A_1) \in C_2(S_2)$.

<u>Theorem 5.3</u>: If $A \subset S_2$, then $\lambda_1(f^{-1}(A)) \subseteq f^{-1}(\lambda_2(A))$.

<u>Proof</u>: $f[\lambda_1(f^{-1}(A))] = \lambda_2(ff^{-1}(A))$ by Property (1) of f,

$= \lambda_2(A)$,

$f[\lambda_1(f^{-1}(A))] = \lambda_2(A)$ implies $\lambda_1(f^{-1}(A)) \subset f^{-1}(\lambda_2(A))$.

<u>Theorem 5.4</u>: If $A \subset S_2$, then $\mu_1(f^{-1}(A)) \subseteq f^{-1}(\mu_2(A))$.

<u>Proof</u>: $f[\mu_1(f^{-1}(A))] = \mu_2(ff^{-1}(A))$ by Property (1) of f,

$= \mu_2(A)$,

$f[\mu_1(f^{-1}(A))] = \mu_2(A)$ implies $\mu_1(f^{-1}(A)) \subset f^{-1}(\mu_2(A))$.

<u>Theorem 5.5</u>: If $X \in L_2(S_2)$, then $f^{-1}(X) \in L_1(S_1)$.

<u>Proof</u>: $\lambda_1(f^{-1}(X)) \subseteq f^{-1}(\lambda_2(X))$ by Theorem 5.3, $f^{-1}(\lambda_2(X)) = f^{-1}(X)$, since $X \in L_2(S_2)$. Hence, $\lambda_1(f^{-1}(X)) \subseteq f^{-1}(X)$. Since $f^{-1}(X) \subseteq \lambda_1(f^{-1}(X))$, we have $\lambda_1(f^{-1}(X)) = f^{-1}(X)$. Therefore, $f^{-1}(X) \in L_1(S_1)$.

<u>Theorem 5.6</u>: If $X \in C_2(S_2)$, then $f^{-1}(X) \in C_1(S_1)$.

<u>Proof</u>: $\mu_1(f^{-1}(X)) \subseteq f^{-1}(\mu_2(X))$ by Theorem 5.4, $f^{-1}(\lambda_2(X)) = f^{-1}(X)$, since $X \in C_2(S_2)$. Therefore, $\mu_1(f^{-1}(X)) \subseteq f^{-1}(X)$. It is always true in an inclusion lattice that $f^{-1}(X) \subseteq \mu_1(f^{-1}(X))$. Hence, $\mu_1(f^{-1}(X)) = f^{-1}(X)$ which implies $f^{-1}(X) \in C_1(S_1)$.

<u>Theorem 5.7</u>: If $X_1 \in L_1(S_1)$, then $g(f(X_1)) \in L_3(S_3)$ where f is a morphism from S_1 to S_2 and g is a morphism from S_2 to S_3.

<u>Proof</u>: By Theorem 5.1, $f(X_1) \in L_2(S_2)$. Now consider $f(X_1)$ replacing X_1 and g replacing f; then we have $g(f(X_1)) \in L_3(S_3)$.

<u>Theorem 5.8</u>: If $X_3 \in L_3(S_3)$, then $f^{-1}(g^{-1}(X_3)) \in L_1(S_1)$, f and g are as defined in Theorem 5.7.

<u>Proof</u>: By Theorem 5.4, $g^{-1}(X_3) \in L_2(S_2)$. Apply Theorem 5.4 again and we have $f^{-1}(g^{-1}(X_3)) \in L_1(S_1)$.

<u>Theorem 5.9</u>: If $X_1 \in C_1(S_1)$, then $g(f(X_1)) \in C_3(S_3)$.

<u>Proof</u>: By Theorem 5.2, $f(X_1) \in C_2(S_2)$. By applying Theorem 5.2 again, we have $g(f(X_1)) \in C_3(S_3)$.

<u>Theorem 5.10</u>: If $X_3 \in C_3(S_3)$, then $f^{-1}(g^{-1}(X_3)) \in C_1(S_1)$.

<u>Proof</u>: By Theorem 5.6, $g^{-1}(X_3) \in C_2(S_2)$. Apply Theorem 5.6 again and we have $f^{-1}(g^{-1}(X_3)) \in C_1(S_1)$.

<u>Theorem 5.11</u>: If $A_3 \subset S_3$, then $\lambda_1(f^{-1}(g^{-1}(A_3))) \subseteq f^{-1}(g^{-1}(\lambda_3(A_3)))$.

<u>Proof</u>: By Theorem 5.4, $\lambda_2(g^{-1}(A_3)) \subseteq g^{-1}(\lambda_3(A_3))$. By Theorem 5.4, we also have $\lambda_1(f^{-1}(g^{-1}(A_3))) \subseteq f^{-1}(\lambda_2(g^{-1}(A_3))) \subseteq f^{-1}(g^{-1}(\lambda_3(A_3)))$.

<u>Theorem 5.12</u>: If $A_3 \subset S_3$, then $\mu_1(f^{-1}(g^{-1}(A_3))) \subseteq f^{-1}(g^{-1}(\mu_3(A_3)))$.

<u>Proof</u>: By Theorem 5.4, $\mu_2(g^{-1}(A_3)) \subseteq g^{-1}(\mu_3(A_3))$. Also by the same theorem, we have $\mu_1(f^{-1}(g^{-1}(A_3))) \subseteq f^{-1}(\mu_2(g^{-1}(A_3))) \subseteq f^{-1}(g^{-1}(\mu_3(A_3)))$.

<u>Theorem 5.13</u>: If $A \subset X_1$ an element of $L_1(S_1)$ and x is a core point of A relative to X_1, then for $f \in \text{Hom}\,(S_1, S_2)$ with Properties (1) and (2), $f(x)$ is a core point of $f(A) \subset f(X_1)$ relative to $f(X_1)$.

<u>Proof</u>: Let c be a core point of A and consider $f(c)$. Let z_1 be any point of $f(X_1)$ such that $z_1 \neq f(c)$; this implies there exists $z \in X_1$ such that $f(z) = z_1$, $z \neq c$. Since c is a core point of A, there exists $w \in \overline{cz}$ such that $\overset{\bullet\ \ \bullet}{\overline{wc}} \subset A$. For all w such that $f(w) \neq f(c)$, $f(w) \in \overline{z_1 f(c)}$ and $\overset{\bullet\qquad\quad\bullet}{\overline{f(w)\, f(c)}} \subset f(A)$. Therefore, $f(c)$ is a core point of $f(A)$ relative to $f(X_1)$. If $f(w) = f(c)$, then $f(\lambda_1(w,c)) = \lambda_2\{f(w), f(c)\} = f(c)$. But $z \in \lambda_1(w,c)$ so $f(w) = f(c) \Longrightarrow f(z) = f(c)$, a contradiction. Therefore, $f(w) \neq f(c)$.

<u>Theorem 5.14</u>: If U_2 is an element of G_{2c} (open sets of S_2), $f^{-1}(U_2)$ is an element of G_{1c} (open sets of S_1).

<u>Proof</u>: Let $z \in f^{-1}(U_2)$, $f(z) \in U_2$; there exists a convex set $U \subset U_2$ such that $f(z)$ is a core point of U, $z \in f^{-1}(U) \subset f^{-1}(U_2)$ and $f^{-1}(U)$ is convex by Theorem 5.6. Let x be any point of S_1 such that $x \neq z$ and $f(x) \neq f(z)$. There exists $w \in \overline{f(x)\, f(z)}$ such that $\overleftrightarrow{w\, f(z)} \subset U$, since $f(z)$ is a core point of U. Let $p \in S_1$, $p \in \overleftrightarrow{xz}$ such that $f(p) = w$; this is possible since $f(\mu_1(z,x)) = \mu_2(f(z), f(x))$, $w \in \overline{f(z)\, f(x)}$, $\overleftrightarrow{pz} \subset f^{-1}(U)$. If $p = z$, then $w = f(p) = f(z)$. If $x \neq z$ and $f(x) = f(z)$, then $f(\lambda_1(z,x)) = \lambda_2(f(z), f(x)) = f(z)$; this implies $\overline{zx} \subset \lambda_1(z,x) \subset f^{-1}(z)$ $\subset f^{-1}(U)$. Therefore, z is a core point of $f^{-1}(U_2)$.

Theorem 5.14 shows the map f is continuous.

<u>Theorem 5.15</u>: If $U_1 \in G_{1c}$ (open sets) relative to a flat X_1, then $f(U_1) \in G_{2c}$ (open sets) relative to $f(X_1)$.

<u>Proof</u>: Let $x \in f(U_1)$; there exists $y \in U_1$ such that $f(y) = x$. Since $U_1 \in G_{1c}$, there exists $U \in G_{1c}$ such that $y \in U \subseteq U_1$. By Theorem 5.13, $f(y) = x$ is a core of $f(U_1)$ relative to $f(X_1)$. By Theorem 5.2, $f(U)$ is convex, since U is convex, $x \in f(U) \subseteq f(U_1)$. Therefore, $f(U_1) \in G_{2c}$.

The above theorem says that f carries open sets into open sets. Hence, f is an open map. It is not true that f is a closed map. To see this, consider the following example.

Let $U_1 = \{(x, 1/x) \mid x > 0\}$ and consider the map g such that $g(x, 1/x) = x$, g has the desired properties of f, but U_1 is closed in the plane, while $g(U_1)$ is open in the image of the plane under g.

Theorem 5.16: If $A \subset S_2$, then $\lambda_1(g^{-1}(A)) = g^{-1}(\lambda_2(A))$, if g is bijective.

Proof: $g(\lambda_1(g^{-1}(A))) = \lambda_2(g(g^{-1}(A)))$ by Property (1) of g

$$= \lambda_2(A)$$

that is $g(\lambda_1(g^{-1}(A))) = \lambda_2(A)$. Apply g^{-1}; we have

$g^{-1}(g(\lambda_1(g^{-1}(A)))) = g^{-1}(\lambda_2(A))$ or $\lambda_1(g^{-1}(A)) = g^{-1}(\lambda_2(A))$, since g is bijective.

Theorem 5.17: If g is bijective, then $g^{-1}(\lambda_2(g(A))) = \lambda_1(A)$, $A \subset S_1$.

Proof: By Property (1) of g, $g(\lambda_1(A)) = \lambda_2(g(A))$. Apply g^{-1}; $g^{-1}(g(\lambda_1(A))) = g^{-1}(\lambda_2(g(A)))$ or $\lambda_1(A) = g^{-1}(\lambda_2(g(A)))$, since g is bijective.

Theorem 5.18: If C_1 is compact, then $f(C_1)$ is compact.

Proof: Let $C_2 = f(C_1)$ and W an open covering of C_2. Let $W^{-1} = \{f^{-1}(U) \mid U \in W\}$. W^{-1} is an open covering of C_1. Since C_1 is compact, there exists a finite set $\widetilde{U}_1, \widetilde{U}_2, \ldots, \widetilde{U}_n$ such that $\widetilde{U}_i \in W^{-1}$, $C_1 \subseteq \bigcup_{i=1}^{n} \widetilde{U}_i$, $\widetilde{U}_i = f^{-1}(U_i)$, $f(C_1) \subseteq \bigcup_{i=1}^{n} f(\widetilde{U}_i)$. Therefore, $f(C_1) = C_2$ is compact.

Theorem 5.18 shows that compact sets are carried into compact sets under f.

CHAPTER VI

SUMMARY

This paper has been concerned with a study of an incidence geometry L(S) of a set S and the related convex sets, order, topological, and uniform structures.

The necessary order structure was developed in Chapter II by use of the idea of a half-space of a hyperflat and the usual order axioms were proved. Use was made in Chapter III of this order structure in conjunction with the related convex sets of the incidence geometry L(S) to prove some of the classical separation theorems.

In Chapter IV, a topology was defined for a set S which admits an incidence geometry such that if S were a real vector space with the standard incidence geometry and collection of convex sets, this topology was the convex core topology. A uniformity was then defined for S and it was shown that in a locally convex topological real vector space the standard uniformity satisfies the conditions of the defined uniformity and the topology thus introduced was sufficient to insure that a real vector space with the standard incidence geometry and order structure is a locally convex topological vector space.

It was shown that the maps in the natural category of general sets with linear and order structures and convex core topologies were open continuous maps which carry compact sets into compact sets.

The following are some questions for further investigation. Could the same results be gotten in Chapter II without the use of Property (12) of C(S)? Could other topologies be characterized as the convex core topology was by the map f? When is the map f a linear map if S is a real vector space?

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